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Integrals of products of Airy functions

J R Albright†

University of Cambridge, Department of Physics, Cavendish Laboratory, Madingley Road, Cambridge CB3 OHE, UK

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Abstract. A large number of indefinite integrals of the form $\int x^n y_1 y_2 \, dx$ have been evaluated in terms of x, y_1, y_2 and their first derivatives; y_1 and y_2 are both solutions of the differential equation $y''=xy$. Some of these integrals can be applied to the quantum mechanical problem of a particle in a uniform field of force.

1. Introduction

Airy functions have been a part of mathematical physics for many years. They may be defined as solutions of the differential equation

$$y''=xy \tag{1}$$

for which it is convenient to write the general solution in the form

$$y=\alpha \mathrm{Ai}(x)+\beta \mathrm{Bi}(x). \tag{2}$$

In this solution α and β are arbitrary constants; $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are Airy functions, chosen conventionally such that $\mathrm{Ai}(x)$ has the property

$$\lim_{x\to\infty} \mathrm{Ai}(x)=0 \tag{3}$$

with $\mathrm{Bi}(x)$ being unbounded in this limit. In this paper we shall use y to designate any linear combination of solutions to (1).

The quantum mechanical problem of a particle in a uniform field of force is described by the Schrödinger equation, which in this case can be readily transformed into (1). Thus Airy functions are effectively the wavefunctions for this problem. The uniform force field has been considered for continuum states (Breit 1928, Mott and Sneddon 1948) and for bound states (Gibbs 1975). In none of these treatments is the normalization of the wavefunction dealt with. For the bound states the normalization requires the integration of the wavefunction over the interval (x_λ, ∞) where x_λ is one of the zeros of the Airy function. The evaluation of such normalization integrals was the problem that led to this work.

If one searches through the usual collections of integrals of transcendental functions (Abramowitz and Stegun 1965, Gradshteyn and Ryzhik 1965, Luke 1962) one finds formulae for integrals of Ai and Bi but not for Ai^2, AiBi or Bi^2. Similarly a search

† Permanent address: Department of Physics, Florida State University, Tallahassee, FL 32306, USA.

through standard references on the properties of such functions (Jeffreys and Jeffreys 1946, Watson 1922) offers no help toward integration of such products.

It is the purpose of this paper to show how these integrals may be performed, with results expressed in terms of functions that are already tabulated.

2. Methods

We shall be concerned with indefinite integrals of the forms

$$\int x^n y^2 \, \mathrm{d}x \tag{4}$$

$$\int x^n y' y \, \mathrm{d}x \tag{5}$$

$$\int x^n y'^2 \, \mathrm{d}x \tag{6}$$

where $n = 0, 1, 2, 3, \ldots$. Since numerical tables of $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, $\mathrm{Bi}(x)$ and $\mathrm{Bi}'(x)$ are readily available (Abramowitz and Stegun 1965, Miller 1971), we shall consider that a form of the indefinite integral will be acceptable if it can be expressed in terms of x, y and y'. To see how this can be done it is useful to examine table 1, in which derivatives of (4), (5) and (6) are systematically displayed. In the construction of table 1 it is essential to make use of (1) to substitute xy for y'' every time the latter appears.

Table 1. Derivatives of products of Airy functions.

	y^2	$y'y$	y'^2	xy^2	$xy'y$	xy'^2	x^2y^2	$x^2y'y$	$x^2y'^2$	x^3y^2	$x^3y'y$
$\mathrm{D}y^2$		2									
$\mathrm{D}y'y$			1	1							
$\mathrm{D}y'^2$					2						
$\mathrm{D}xy^2$	1				2						
$\mathrm{D}xy'y$		1				1	1				
$\mathrm{D}xy'^2$			1					2			
$\mathrm{D}x^2y^2$				2				2			
$\mathrm{D}x^2y'y$					2				1	1	
$\mathrm{D}x^2y'^2$						2					2

The property of table 1 that renders tractable the integrations in question is that one can find sets of rows for which the elements appear only in small numbers of columns for which the integrals are not already known. There are two trivial examples of this, the rows labelled $\mathrm{D}y^2$ and $\mathrm{D}y'^2$. A more instructive example is the set with $\mathrm{D}y'^2$ and $\mathrm{D}xy'^2$. By subtracting these two rows one obtains the equation

$$\mathrm{D}(xy^2 - y'^2) = y^2 \tag{7}$$

which is exactly what is wanted for the normalization integral.

The rows corresponding to $Dy'y$, Dxy'^2 and Dx^2y^2 form another such set, with the equation

$$D\begin{pmatrix} y'y \\ xy'^2 \\ x^2y^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} y'^2 \\ xy^2 \\ x^2y'y \end{pmatrix}. \tag{8}$$

Inversion of the 3×3 matrix yields

$$\begin{pmatrix} y'^2 \\ xy^2 \\ x^2y'y \end{pmatrix} = D\tfrac{1}{6}\begin{pmatrix} 4 & 2 & -2 \\ 2 & -2 & 2 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} y'y \\ xy'^2 \\ x^2y^2 \end{pmatrix} \tag{9}$$

from which it is an easy matter to read off three expressions for integrals of products of Airy functions.

General recursion formulae for arbitrary n can be obtained in this way, since

$$D\begin{pmatrix} x^ny'y \\ x^{n+1}y'^2 \\ x^{n+2}y^2 \end{pmatrix} = \begin{pmatrix} nx^{n-1}y'y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ n+1 & 0 & 2 \\ 0 & n+2 & 2 \end{pmatrix}\begin{pmatrix} x^ny'^2 \\ x^{n+1}y^2 \\ x^{n+2}y'y \end{pmatrix}. \tag{10}$$

This set can be solved, since in principle the quantity

$$\int x^{n-1}y'y\,dx$$

is already known. The results of the solution of this set of equations are in the appendix.

A simpler form for

$$I = \int x^ny'y\,dx \tag{11}$$

is available from integration by parts, since

$$I = x^ny^2 - \int nx^{n-1}y\,dx - I$$

$$I = \tfrac{1}{2}\left(x^ny^2 - n\int x^{n-1}y^2\,dx\right). \tag{12}$$

Results can also be derived for integrals of the form

$$\int x^ny_1y_2\,dx \qquad \int x^ny_1'y_2\,dx \qquad \int x^ny_1'y_2'\,dx \tag{13}$$

where y_1 and y_2 are any two solutions of (1). The first of these can be evaluated by writing the identity

$$\int x^ny_1y_2\,dx = \tfrac{1}{2}\left(\int x^n(y_1+y_2)^2\,dx - \int x^ny_1^2\,dx - \int x^ny_2^2\,dx\right). \tag{14}$$

Each of the integrals on the right-hand side can be evaluated by using (A.12), and the result is displayed in (A.22).

The third expression in (13) is evaluated in a way that is entirely analogous to that just used.

To evaluate the second expression in (13) it is helpful to use the Wronskian determinant:

$$W = y_1 y_2' - y_1' y_2. \tag{15}$$

Since y_1 and y_2 are solutions to (1) it follows that $W' = 0$, the Wronskian is constant. Next one writes two equations

$$\int x^n (y_1' y_2 + y_1 y_2')\,\mathrm{d}x = \int x^n \mathrm{D}(y_1 y_2)\,\mathrm{d}x \tag{16}$$

$$\int x^n (y_1' y_2 - y_1 y_2')\,\mathrm{d}x = -\int x^n W\,\mathrm{d}x = -(n+1)^{-1} x^{n+1} W. \tag{17}$$

Adding (16) and (17) yields

$$2\int x^n y_1' y_2\,\mathrm{d}x = \int x^n \mathrm{D}(y_1 y_2)\,\mathrm{d}x - (n+1)^{-1} x^{n+1} W.$$

The first term on the right-hand side can be integrated by parts; the result is (A.24).

3. Results and conclusions

The results are collected in the appendix. Like most results in integral calculus, they are easier to check than to obtain. Since the results involving $y_1 y_2$ were derived without regard to which solutions of (1) are intended, they are therefore more general than those involving y^2; the latter can be derived readily from the former.

An outcome of these formulae is that the wavefunction for a particle in a uniform force field can have its normalization constant expressed in terms of tabulated quantities. Further, since the integrals are indefinite, it is possible to calculate probabilities of measurement of the position of the particle for any desired interval. Also, diagonal matrix elements of position, of momentum or combinations of both can readily be obtained. Unfortunately the method used here is not suited for the calculation of off-diagonal elements, since the matrix of derivatives cannot be partitioned in the way it was done here.

Appendix

Integrals of products of Airy functions are collected here; y refers to any linear combination of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$; n is a positive integer.

$$\int y^2\,\mathrm{d}x = xy^2 - y'^2 \tag{A.1}$$

$$\int y'y\,\mathrm{d}x = \tfrac{1}{2}y^2 \tag{A.2}$$

$$\int y'^2\,\mathrm{d}x = \tfrac{1}{3}(2y'y + xy'^2 - x^2y^2) \tag{A.3}$$

$$\int xy^2\,\mathrm{d}x = \tfrac{1}{3}(y'y - xy'^2 + x^2y^2) \tag{A.4}$$

$$\int xy'y\,\mathrm{d}x = \tfrac{1}{2}y'^2 \tag{A.5}$$

$$\int xy'^2\,\mathrm{d}x = \tfrac{1}{5}[3(xy'y - \tfrac{1}{2}y^2) + x^2y'^2 - x^3y^2] \tag{A.6}$$

$$\int x^2y^2\,\mathrm{d}x = \tfrac{1}{5}[2(xy'y - \tfrac{1}{2}y^2) - x^2y'^2 + x^3y^2] \tag{A.7}$$

$$\int x^2y'y\,\mathrm{d}x = \tfrac{1}{6}(x^2y^2 - 2y'y + 2xy'^2) \tag{A.8}$$

$$\int x^2y'^2\,\mathrm{d}x = \tfrac{1}{7}(4x^2y'y - 4y'^2 + x^3y'^2 - x^4y^2) \tag{A.9}$$

$$\int x^3y^2\,\mathrm{d}x = \tfrac{1}{7}(3x^2y'y - 3y'^2 - x^3y'^2 + x^4y^2) \tag{A.10}$$

$$\int x^3y'y\,\mathrm{d}x = \tfrac{1}{5}(-3xy'y + \tfrac{3}{2}y^2 + \tfrac{3}{2}x^2y'^2 + x^3y^2) \tag{A.11}$$

$$\int x^ny^2\,\mathrm{d}x = \frac{1}{2n+1}\left(nx^{n-1}y'y - n(n-1)\int x^{n-2}y'y\,\mathrm{d}x - x^ny'^2 + x^{n+1}y^2\right) \tag{A.12}$$

$$\int x^ny'y\,\mathrm{d}x = \tfrac{1}{2}\left(x^ny^2 - n\int x^{n-1}y^2\,\mathrm{d}x\right) \tag{A.13}$$

$$\int x^ny'y\,\mathrm{d}x = \frac{1}{2n-1}\Bigg[-\tfrac{1}{2}n(n-1)\left(x^{n-2}y'y - (n-2)\int x^{n-3}y'y\,\mathrm{d}x\right) + \tfrac{1}{2}nx^{n-1}y'^2 + \tfrac{1}{2}(n-1)x^ny^2\Bigg] \tag{A.14}$$

$$\int x^ny'^2\,\mathrm{d}x = \frac{1}{2n+3}\left[(n+2)\left(x^ny'y - n\int x^{n-1}y'y\,\mathrm{d}x\right) + x^{n+1}y'^2 - x^{n+2}y^2\right] \tag{A.15}$$

$$\int y_1y_2\,\mathrm{d}x = xy_1y_2 - y_1'y_2' \tag{A.16}$$

$$\int y_1'y_2\,\mathrm{d}x = \tfrac{1}{2}(y_1y_2 + xy_1'y_2 - xy_1y_2') \tag{A.17}$$

$$\int y_1'y_2'\,\mathrm{d}x = \tfrac{1}{3}(y_1'y_2 + y_1y_2' + xy_1'y_2' - x^2y_1y_2) \tag{A.18}$$

$$\int xy_1y_2\,\mathrm{d}x = \tfrac{1}{6}(y_1'y_2 + y_2y_2' - 2xy_1'y_2' + 2x^2y_1y_2) \tag{A.19}$$

$$\int xy_1'y_2\,dx = \tfrac{1}{4}(2y_1'y_2' + x^2y_1'y_2 - x^2y_1y_2') \tag{A.20}$$

$$\int xy_1'y_2'\,dx = \tfrac{1}{5}[\tfrac{3}{2}(xy_1'y_2 + xy_1y_2' - y_1y_2) + x^2y_1'y_2' - x^3y_1y_2] \tag{A.21}$$

$$\int x^ny_1y_2\,dx = \frac{1}{2(2n+1)}\Big(nx^{n-1}(y_1'y_2 + y_1y_2') - 2x^ny_1'y_2' + 2x^{n+1}y_1y_2 - n(n-1)\int x^{n-2}(y_1'y_2 + y_1y_2')\,dx\Big) \tag{A.22}$$

$$\int x^ny_1y_2\,dx = \tfrac{1}{2}\Big(x^{n-1}(y_1'y_2 + y_1y_2') - (n-1)\int x^{n-2}(y_1'y_2 + y_1y_2')\,dx - 2\int x^{n-1}y_1'y_2'\,dx\Big), \qquad n \geqslant 1 \tag{A.23}$$

$$\int x^ny_1'y_2\,dx = \tfrac{1}{2}\Big(x^ny_1y_2 - n\int x^{n-1}y_1y_2\,dx + (n+1)^{-1}x^{n+1}(y_1'y_2 - y_1y_2')\Big) \tag{A.24}$$

$$\int x^ny_1'y_2'\,dx = \frac{1}{2(2n+3)}\Big[(n+2)\Big(x^n(y_1'y_2 + y_1y_2') - n\int x^{n-1}(y_1'y_2 + y_1y_2')\,dx\Big) + 2x^{n+1}(y_1'y_2' - xy_1y_2)\Big]. \tag{A.25}$$

Integrals (A.16) through (A.25) are true for y_1 and y_2 being any solutions of (1). If $y_1 = \mathrm{Ai}(x)$ and $y_2 = \mathrm{Bi}(x)$ using the customary definitions (Abramowitz and Stegun 1965, Miller 1971) some of the integrals can be simplified by means of the Wronskian relation

$$\mathrm{Ai}\,\mathrm{Bi}' - \mathrm{Ai}'\,\mathrm{Bi} = \pi^{-1}.$$

For example,

$$\int \mathrm{Ai}'\,\mathrm{Bi}\,dx = \tfrac{1}{2}(\mathrm{Ai}\,\mathrm{Bi} - x/\pi). \tag{A.17a}$$

(A.20) and (A.24) can both be simplified in this way.

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